3RD DEGREE POLYNOMIALS & EQUATIONS

GRADE 12

(but Grade 11's are invited!)

It is essential to do EVERY question if you want to be an expert on "cubics"!

1. Observe the following solution of a *quadratic* equation:

$$x^2 - x - 6 = 0$$
$$\therefore\ (x - 3)(x + 2) = 0$$
$$\therefore\ x - 3 = 0 \quad \textbf{OR} \quad x + 2 = 0$$
$$\therefore\ x = 3 \qquad\qquad \therefore\ x = -2$$

1.1 Write down the *factors* of the *expression* $x^2 - x - 6$. (1)

1.2 Write down the *roots* of the *equation* $x^2 - x - 6 = 0$. (1)

1.3 Make a conjecture about what you notice when comparing 1.1 and 1.2. (1)

1.4 If $f(x) = x^2 - x - 6$, find the value of $f(3)$ and $f(-2)$. (2)

1.5 Make a conjecture about what you notice when comparing 1.2 and 1.4. Explain. (2)

2.1 Write down a *cubic* equation $g(x) = 0$ with roots -4, -2 and 1 - with $g(x)$ in factorised form. (2)

2.2.1 Write down the roots of the cubic equation: $(x + 1)(2x - 1)(x - 4) = 0$, (3)

2.2.2 If $f(x) = (x + 1)(2x - 1)(x - 4)$, determine the values of $f(-1)$, $f\left(\frac{1}{2}\right)$, and $f(4)$. (1)

3. Complete the following:

3.1 If $x - 5$ is a factor of our expression $f(x)$, then

3.1.1 5 is a . . . . . of $f(x) = 0$ (the equation), and (1)

3.1.2 the value of $f(5)$ is . . . . . (1)

3.2 If -1 is a root of a (quadratic or cubic) equation $g(x) = 0$, then . . . . . is a factor of the expression $g(x)$. (1)

3.3.1 If p is a root of an equation $f(x) = 0$, then $f(p) =$ . . . . . (1)

3.3.2 If $-q$ is a root of an equation $g(x) = 0$, then $g(-q) =$ . . . . . (1)

3.4 If 2 is a root of an equation $f(x) = 0$, then $f(2) =$ . . . . and . . . . . is a factor of $f(x)$. (2)

3.5 If r is a root of an equation $f(x) = 0$, then $f(r) =$ . . . . and . . . . . is a factor of $f(x)$. (2)

4.1 If $x - 2$ is a factor of $f(x) = x^3 - 12x + 16$, find the other factors of $f(x)$. (3)

4.2 Now solve the equation $x^3 - 12x + 16 = 0$ (3)

5. Given: $f(x) = 2x^3 - x^2 - 13x - 6$

5.1 Determine $f(3)$. (2)

5.2 Explain the meaning of your answer in 5.1. (2)

5.3 Factorise $f(x)$ completely. (3)

5.4 Solve for x if $f(x) = 0$. (2)

6. Given: $f(x) = x^3 + 3x^2 - 2x - 4$

6.1 Prove that $(x + 1)$ is a factor of $f(x)$. (2)

6.2 Now solve for x in $x^3 + 3x^2 - 2x - 4 = 0$. (5)

7.1.1 By referring to question 1, draw a rough sketch of the graph of $y = x^2 - x - 6$. (You do not need to show the turning point.) (3)

7.1.2 Complete: The roots of the equation $x^2 - x - 6 = 0$ are also the . . . . . of the parabola, $y = x^2 - x - 6$. (1)

7.2.1 Also sketch the graph of $y = (x - 1)^2$. (2)

7.2.2 This parabola has only one x-intercept which is also the . . . . . of the parabola. (1)

Read about the shape of a cubic graph before continuing . . .

8.1.1 $f(x) = x^3 - 4x$
Factorise $f(x)$ completely. (2)

8.1.2 What are the x-intercepts of the graph of f? (2)

8.1.3 Write down the y-intercept of the graph of f. (1)

8.1.4 Draw a sketch of f, indicating these intercepts. (3)

9. Given the function defined by $f(x) = x^3 - 3x^2 + 4$:

9.1 Calculate $f(-1)$, and hence solve the equation $f(x) = 0$. (5)

9.2 Draw a sketch of the graph, f, indicating only the intercepts with the axes. (4)

10. Solve the following cubic equations:

10.1 $x^3 + 6x^2 = 0$	10.2 $x^3 + x^2 - x - 1 = 0$
10.3 $x^3 + 2x^2 - 3x = 0$	10.4 $-x^3 + 4x^2 - 4x = 0$
10.5 $x^3 - 12x + 16 = 0$	10.6 $x^3 + 4x^2 + x - 6 = 0$
10.7 $x^3 - x^2 - x + 10 = 0$	10.8 $-x^3 + 2x^2 + x - 2 = 0$
10.9 $-3x^3 + 4x^2 + x - 2 = 0$	10.10 $x^3 + 4x^2 + x - 6 = 0$
10.11 $-2x^3 + 7x^2 + 5x - 4 = 0$	10.12 $2x^3 - 7x^2 + 4x + 4 = 0$

(3) (3×5) (6×6)

SHAPE OF A CUBIC GRAPH

Plot some points to see the shape of the graphs $y = x^3$ and $y = -x^3$. You will then understand that . . .

The shape of a cubic graph $y = ax^3$ . . . , where $a > 0$ is 'roughly' 

but $y = ax^3$, where $a < 0$ is 'roughly' 

In CALCULUS you will acquire a new 'tool', called the DERIVATIVE . . . You will find out:

◈ **WHAT IT IS**
- *Simply . . .* It is the **GRADIENT** of a graph *(See below for more details)*
- *Mathematically . . .* You will get used to the limit concept behind the definition - no rush for this!

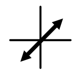
◈ **HOW TO FIND IT**
- You will find it *by rules* - simple ones!
- *Mathematically . . .* You will, in time, get used to finding it "from first principles".

◈ **WHERE TO USE IT**
- *Graphs* e.g. **The cubic graph!**
- *Applications* - mainly to find minimum & maximum values

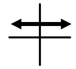
The DERIVATIVE of a function is:

▣ ***the GRADIENT of the function . . .***

e.g. $y = x$, i.e. $f(x) = x$ $y = 2$, i.e. $f(x) = 2$



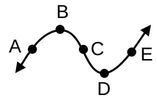
The derivative is the gradient, m = 1



The derivative is the gradient, m = 0

The gradient of a line is **constant** (i.e., the same at all pts.)

▣ ***The derivative is the gradient of a function AT A POINT . . .***

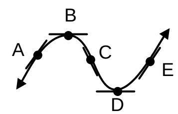


The DERIVATIVE/GRADIENT is
- positive at A and E
- **zero at B & D!!!**
- negative at C

and not constant!

"The DERIVATIVE is zero at the turning points" *is the 'fact' that we will use to find the turning points of our cubic graphs.*

Finally, drawing "little tangents" at A, B, C, D and E makes it easier to picture the ***gradient of the curve*** at these points . . . it is ***the gradient of the tangent at the point***.



▣ ***Finally, the DERIVATIVE is . . .***
The gradient of the tangent to a curve at a point!

3RD DEGREE POLYNOMIALS & EQUATIONS

1.1 The *factors* of the *expression* are $(x - 3)$ and $(x + 2)$. ◂

1.2 The *roots* of the *equation* are 3 and - 2. ◂

1.3 If the **factors** are $(x - p)$ and $(x + q)$ . . . *of the expression*
then the **roots** are p and - q . . . *of the equation.*

1.4 $f(\mathbf{3}) = \mathbf{0}$ and $f(\mathbf{-2}) = \mathbf{0}$ ◂

1.5 The 'root' values of x make the expression, **f(x)**, zero,
i.e. they satisfy the equation **f(x) = 0**. . . . *they make it "true"*

2.1 $(x + 4)(x + 2)(x - 1) = 0$ ◂

2.2.1 $x = -1$ or $\frac{1}{2}$ or 4 ◂

2.2.2 $f(-1) = 0$; $f\left(\frac{1}{2}\right) = 0$ and $f(4) = 0$ ◂

[Note : $(x + 1)$, $(2x - 1)$ & $(x - 4)$ are *factors* of $f(x)$
-1, $\frac{1}{2}$ & 4 are *roots* of $f(x) = 0$]

3.1.1 root ◂ 3.1.2 0 ◂

3.2 $x + 1$ ◂

3.3.1 $f(p) = \mathbf{0}$ ◂ 3.3.2 $g(-q) = \mathbf{0}$ ◂

3.4 $f(2) = \mathbf{0}$ and **x - 2** is a factor of $f(x)$

3.5 $f(r) = \mathbf{0}$ and **x - r** is a factor of $f(x)$

4.1 $x - 2$ a factor of $f(x)$ implies that

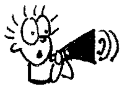
$x^3 - 12x + 16 = (x - 2)(\quad ? \quad)$ ← No term in x^2

$= (x - 2)(x^2 \ldots\ldots - 8)$ $[-2x^2 + \mathbf{2x^2} = 0]$

$= (x - 2)(x^2 + 2x - 8)$ Check: ✓ $-4x - 8x = -12x$

$= (x - 2)(x + 4)(x - 2)$

$= \mathbf{(x - 2)^2(x + 4)}$ ◂ - the **FACTORS**



4.2 The equation : $x^3 - 12x + 16 = 0$
$(x - 2)^2(x + 4) = 0$
$\therefore x = \mathbf{2}$ or $\mathbf{-4}$ ◂ - the **ROOTS**

5.1 $f(x) = 2x^3 - x^2 - 13x - 6$
$\therefore f(3) = 2(3)^3 - 3^2 - 13(3) - 6$
$= 54 - 9 - 39 - 6$
$= \mathbf{0}$ ◂

5.2 $f(3) = 0$ means that :
3 is a root of the equation $f(x) = 0$
and $x - 3$ is a factor of the expression $f(x)$.

5.3 $\therefore 2x^3 - x^2 - 13x - 6 = (x - 3)(2x^2 \ldots\ldots + 2)$ ← $+5x^2$
$= (x - 3)(2x^2 + 5x + 2)$ Check: ✓ $-15x + 2x = -13x$
$= \mathbf{(x - 3)(2x + 1)(x + 2)}$ ◂

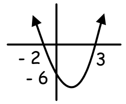
5.4 $2x^3 - x^2 - 13x - 6 = 0$
$\therefore (x - 3)(2x + 1)(x + 2) = 0$
$x = \mathbf{3}$ or $\mathbf{-\frac{1}{2}}$ or $\mathbf{-2}$ ◂ . . . *the* ***roots*** *of the equation*

6.1 $f(x) = x^3 + 3x^2 - 2x - 4$
$f(-1) = (-1)^3 + 3(-1)^2 - 2(-1) - 4$
$= -1 + 3 + 2 - 4$
$= 0$
[∴ - 1 is a root of the equation f(x) = 0]
$\therefore x + 1$ is a factor of the expression $f(x)$ ◂

6.2 $\therefore x^3 + 3x^2 - 2x - 4 = (x + 1)(x^2 \ldots\ldots - 4)$
$= (x + 1)(x^2 \mathbf{+ 2x} - 4)$ ← *no further factors !*

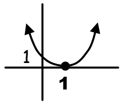
$f(x) = 0 \Rightarrow (x + 1)(x^2 + 2x - 4) = 0$
$\therefore$ **Either** $x + 1 = 0$ **or** $x^2 + 2x - 4 = 0$
$\therefore x = \mathbf{-1}$ ◂ $\therefore x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-4)}}{2(1)}$
$= \frac{-2 \pm \sqrt{20}}{2}$
$\approx \mathbf{1{,}24}$ or $\mathbf{-3{,}24}$ ◂

7.1.1 x-intercepts (roots) : $x = 3$ or -2
Y-intercept *(when x = 0)* : $y = -6$



7.1.2 The ROOTS of the equation $x^2 - x - 6 = 0$ are also the ROOTS (x-intercept) of the parabola $y = x^2 - x - 6$.

7.2.1



[Note : $\mathbf{y = (x - 1)^2}$ is also
$y = x^2 - 2x + 1$
- the trinomial is a **PERFECT SQUARE !**
∴ **only one root**]

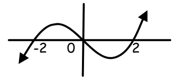
7.2.2 The x-intercept is also the **turning point** of the parabola.

8.1.1 $f(x) = x(x^2 - 4)$
$= \mathbf{x(x + 2)(x - 2)}$ ◂

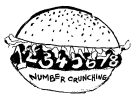
8.1.2 $x = \mathbf{0}$ or $\mathbf{-2}$ or $\mathbf{2}$ ◂

8.1.3 $\mathbf{y = 0}$ ◂

8.1.4



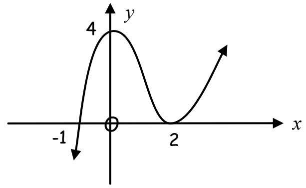
9.1 $f(x) = x^3 - 3x^2 + 4$
$\therefore f(-1) = (-1)^3 - 3(-1)^2 + 4$
$= -1 - 3 + 4$
$= \mathbf{0}$ ◂



$\therefore x + 1$ is a factor of $f(x)$
$\therefore f(x) = (x + 1)(\quad ? \quad)$
$= (x + 1)(x^2 \ldots\ldots + 4)$ $+x^2 \mathbf{- 4x^2} = -3x^2$
$= (x + 1)(x^2 - 4x + 4)$ $+ 4x - 4x = 0$ ✓
$= (x + 1)(x - 2)^2$

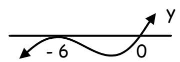
$\therefore (x + 1)(x - 2)^2 = 0$
$\therefore x = \mathbf{-1}$ or $\mathbf{2}$ ◂

9.2

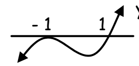


Calculus (next section) will enable you to find the turning points of this cubic graph!

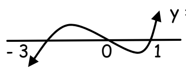
10.1 $x^2(x + 6) = 0$
$\therefore x = \mathbf{0}$ or $\mathbf{-6}$ ◂



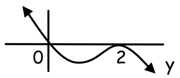
10.2 $x^2(x + 1) - (x + 1) = 0$
$\therefore (x + 1)(x^2 - 1) = 0$ . . . *grouping*
$\therefore (x + 1)(x + 1)(x - 1) = 0$
$\therefore (x + 1)^2(x - 1) = 0$
$\therefore x = \pm\mathbf{1}$ ◂



10.3 $x(x^2 + 2x - 3) = 0$
$\therefore x(x + 3)(x - 1) = 0$
$\therefore x = \mathbf{0}$ or $\mathbf{-3}$ or $\mathbf{1}$ ◂



10.4 $-x^3 + 4x^2 - 4x = 0$
$\times(-1)$ $\therefore x^3 - 4x^2 + 4x = 0$
$\therefore x(x^2 - 4x + 4) = 0$
$\therefore x(x - 2)^2 = 0$
$\therefore x = \mathbf{0}$ or $x = \mathbf{2}$ ◂



10.5 Let $f(x) = x^3 - 12x + 16$

Try $f(1) = 1 - 12 + 16 \neq \mathbf{0}$

$f(-1) = -1 + 12 + 16 \neq \mathbf{0}$

$\mathbf{f(2)} = 8 - 24 + 16 = \mathbf{0}$ **!**

$\therefore$ $\mathbf{x - 2}$ is a factor of $f(x)$

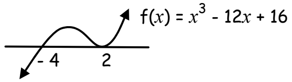
$\therefore$ $f(x) = \mathbf{(x - 2)(\quad ? \quad)}$

$= (x - 2)(x^2 \ldots\ldots - 8)$ $\quad \ldots \left(2x^2 + \mathbf{2x^2} = 0\right)$

$= (x - 2)(x^2 + 2x - 8)$ $\quad \ldots \left(-4x - 8x = -12x \checkmark\right)$

$= (x - 2)(x + 4)(x - 2)$

$= (x - 2)^2(x + 4)$



$\therefore$ Solution to $f(x) = 0$: $\mathbf{x = 2}$ **or** $\mathbf{-4}$ ◄ ... (the roots)

10.6 Let $f(x) = x^3 + 4x^2 + x - 6$

$\mathbf{f(1)} = 1 + 4 + 1 - 6 = \mathbf{0}$ **!**

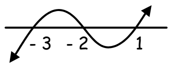
$\therefore$ $x - 1$ is a factor of $f(x)$

$\therefore$ $f(x) = (x - 1)(x^2 \ldots\ldots + 6)$ $\quad -x^2 + \mathbf{5x^2} = 4x^2$

$= (x - 1)(x^2 + 5x + 6)$ $\quad$ (Check: $-5x + 6x = -x$ ✓)

$= (x - 1)(x + 2)(x + 3)$

$\therefore$ Soln. to $f(x) = 0$: $\mathbf{x = 1}$ **or** $\mathbf{-2}$ **or** $\mathbf{-3}$ ◄



10.7 Let $f(x) = x^3 - x^2 - x + 10$

I can see $f(1) \neq 0$, $f(-1) \neq 0$, $f(2) \neq 0$, ... maybe $f(-2)$?

$\mathbf{f(-2)} = -8 - 4 + 2 + 10 = \mathbf{0}$ **!**

$\therefore$ $x + 2$ is a factor of $f(x)$

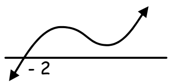
$\therefore$ $f(x) = (x + 2)(x^2 - 3x + 5)$ $\quad \ldots$ (Check: $-6x + 5x = -x$ ✓)

$\therefore$ Solution to $f(x) = 0$:

$\mathbf{x = -2}$ ◄ **or** $x = \dfrac{-(-3) \pm \sqrt{(-3)2 - 4(1)(5)}}{2(1)}$

$= \dfrac{3 \pm \sqrt{\mathbf{-11}}}{2}$ ← $\sqrt{\textbf{neg. no}}$ is non-real !

invalid



Only 1 x-intercept !

10.8 Let $f(x) = -x^3 + 2x^2 + x - 2$

$\mathbf{f(1)} = -1 + 2 + 1 - 2 = \mathbf{0}$

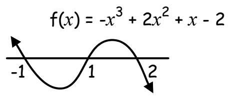
$\therefore$ $x - 1$ is a factor of $f(x)$

$\therefore$ $f(x) = (x - 1)(-x^2 \ldots\ldots + 2)$

$= (x - 1)\mathbf{(-x^2 + x + 2)}$

$= \mathbf{-}(x - 1)\mathbf{(x^2 - x - 2)}$

$= -(x - 1)(x - 2)(x + 1)$



$f(x) = 0$ ➔ $-(x - 1)(x - 2)(x + 1) = 0$

$\therefore$ $(x - 1)(x - 2)(x + 1) = 0$

$\therefore$ $\mathbf{x = 1}$ **or** $\mathbf{2}$ **or** $\mathbf{-1}$ ◄

10.9 Let $f(x) = 3x^3 - 4x^2 - x + 2$

$\mathbf{f(1)} = 3 - 4 - 1 + 2 = 5 - 5 = \mathbf{0}$ **!**

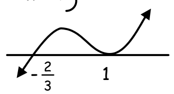
$\therefore$ $x - 1$ is a factor

$\therefore$ $f(x) = (x - 1)(3x^2 \ldots\ldots - 2)$

$= (x - 1)(3x^2 \mathbf{- x} - 2)$ $\quad \ldots$ (Check: $x - 2x = -x$ ✓)

$= (x - 1)(3x + 2)(x - 1)$

$= (x - 1)^2(3x + 2)$



$\therefore$ Solution to $f(x) = 0$: $\mathbf{x = 1}$ **or** $\mathbf{-\frac{2}{3}}$ ◄

10.10 Let $f(x) = x^3 + 4x^2 + x - 6 = 0$

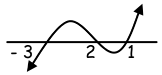
$\mathbf{f(1)} = 1 + 4 + 1 - 6 = \mathbf{0}$

$\therefore$ $x - 1$ is a factor of $f(x)$

$\therefore$ $f(x) = (x - 1)(x^2 \ldots + 6)$

$= (x - 1)(x^2 + 5x + 6)$

$= (x - 1)(x + 2)(x + 3)$



$f(x) = 0$ ➔ $(x - 1)(x + 2)(x + 3) = 0$

$\therefore$ $\mathbf{x = 1}$ **or** $\mathbf{-2}$ **or** $\mathbf{-3}$ ◄

10.11 Let $f(x) = -2x^3 + 7x^2 + 5x - 4 = 0$

$f(1) \neq 0$

$\mathbf{f(-1)} = +2 + 7 - 5 - 4 = \mathbf{0}$

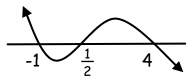
$\therefore$ $(x + 1)$ is a factor of $f(x)$

$\therefore$ $f(x) = (x + 1)(-2x^2 \ldots - 4)$

$= (x + 1)(-2x^2 + 9x - 4)$

$= -(x + 1)(2x^2 - 9x + 4)$

$= -(x + 1)(2x - 1)(x - 4)$



$f(x) = 0$ ➔ $-(x + 1)(2x - 1)(x - 4) = 0$

$\therefore$ $(x + 1)(2x - 1)(x - 4) = 0$

$\therefore$ $\mathbf{x = -1}$ **or** $\mathbf{\frac{1}{2}}$ **or** $\mathbf{4}$ ◄

10.12 Let $f(x) = 2x^3 - 7x^2 + 4x + 4$

$\mathbf{f(2)} = 16 - 28 + 8 + 4 = \mathbf{0}$ **!**

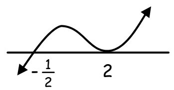
Clearly $f(1) \neq 0$ & $f(-1) \neq 0$

$\therefore$ $f(x) = (x - 2)(2x^2 \ldots\ldots - 2)$

$= (x - 2)(2x^2 - 3x - 2)$

$= (x - 2)(2x + 1)(x - 2)$

$= (x - 2)^2(2x + 1)$



$\therefore$ Solution to $f(x) = 0$: $\mathbf{x = 2}$ **or** $\mathbf{-\frac{1}{2}}$ ◄